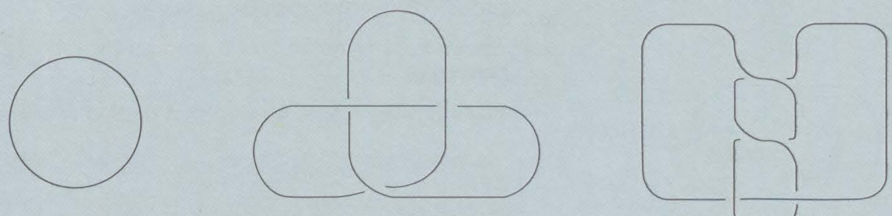
Colouring KNOTS

by Wee Hoe Teck & Wong Yan Loi

Introduction

A knot is a circle, possibly knotted in Euclidean 3-space. It is obtained by joining the two ends of a piece of knotted string together. In figure 1, examples of knots such as the unknot, the trefoil knot and the figure 8 knot are shown. Generally speaking, knot theory is the study of how such objects sit in 3-space.

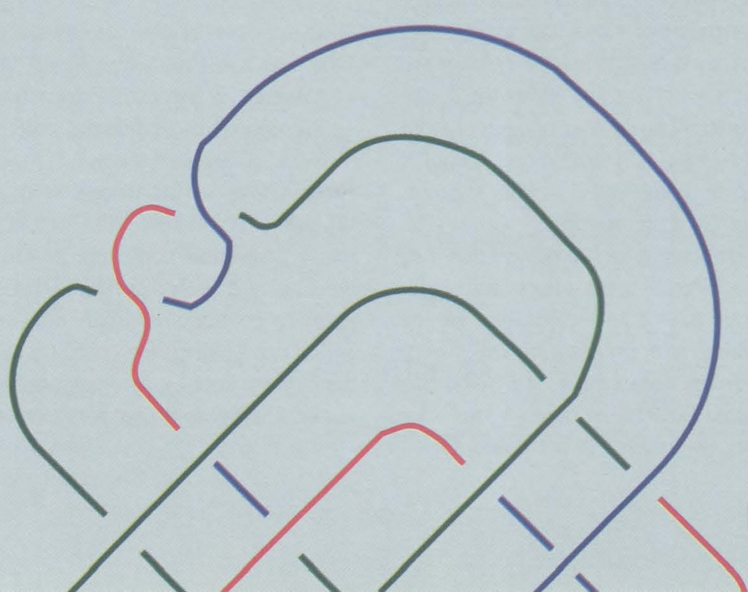


Unknot Trefoil Figure 8

Figure 1

The first analytical study of knots was probably C. F. Gauss' investigation of electrodynamics [3] in 1877. Gauss studied the effect of a knotted electric wire L_1 on another wire L_2. He discovered that the total electric potential energy induced on L_2 due to the magnetic field created by L_1 depends only on the so called linking number of the two wires.

The study of knots is closely related to the study of other geometric objects such as curves and surfaces [2], [6]. Recently its impact and connection with physics [1], [4] and DNA



research [7] cause much interest and attention among mathematicians. A primary problem in knot theory is to distinguish different knots. For this purpose, two knots are considered to be same or equivalent if one can be deformed, pulled or stretched but without cutting the string, to the other. An equivalent definition of this notion will be given in section 2. For instance, the two knots in figure 2 are considered to be the same since they both represent the trefoil knot. However the figure 8 knot can be shown to be different.

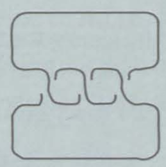
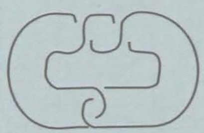


Figure 2



There are various approaches to the problem of distinguishing knots. One way is to look for invariants of knots. Basically, if a knot is found to possess a certain property which remains unchanged when it is deformed to another shape, then such a property defines a knot invariant. The point is that two knots having distinct invariants are different. For example, the crossing number of a knot which is the minimum number of crossings that the knot can have when it is laid on a plane is a knot invariant. The trefoil has crossing number 3 whereas the figure-8 knot has crossing number 4. Classically, there are various knot invariants such as the unknotting number, the bridge number and many of the knot polynomials [6].

In this paper, the colourability of knots is discussed. One attempts to colour the arcs or the overpasses of a knot diagram by using 3 colours such that at least 2 colours are used and at each crossing, either the 3 arcs are coloured by one colour or each of them is coloured by a different colour. A knot K is said to be colourable if a knot diagram of K can be coloured by 3 colours as described above. It can be proved that if K_1 and K_2 are equivalent knots and K_1 is colourable, then K_2 is colourable. Hence colourability is a property of knots or a knot invariant. For example, the trefoil can be coloured but not the figure-8 knot. By definition, the unknot is not colourable.

The idea of colouring knots was first considered by R. Fox. In [2], Fox showed that the colourability of a knot is related to the existence of a so called 3-fold irregular branched covering of the 3-dimensional sphere branched along the knot. Therefore in order to understand such objects, it is necessary to investigate which knots are colourable. The concept of colourability has a natural generalization to p-colourability with $p \geq 3$.

The investigation of this subject is part of the project taken by the first author in his participation of the Science Research Programme 1994. In [8], an elementary proof of a necessary and sufficient condition for doubled knots to be p-colourable as well as an algorithm to colour p-colourable doubled knots are presented.

Colourability

A knot is a closed loop in the Euclidean 3-space. A diagram for a knot K is a projection of K onto the plane such that all self-intersections are non-tangential and are double points only. Gaps are left in the diagram to indicate which parts of the knot pass under other parts. See the examples in figure 1. Two knots are said to be equivalent if loosely speaking, one can be continuously deformed to the other. To avoid the notion of continuity, there is a combinatorial approach due to Reidemeister [6].

Definition

The following moves applied to a portion of a knot diagram are called Reidemeister moves.

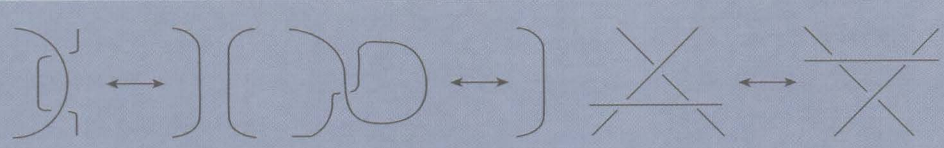


Figure 3

Theorem

Two knots are equivalent if and only if their diagrams can be transformed to each other by a finite sequence of Reidemeister moves.

The two knot diagrams in figure 2 both represent the trefoil knot and they can be transformed to each other by applying Reidemeister moves. Inequivalent knots can be distinguished by many methods. One way is to use the colourability of knots.

Definition

Let three colours be given. A knot diagram is called colourable if each arc can be drawn using one of the colours such that

(i) at least two of the colours are used,
(ii) at any crossing either the 3 arcs are coloured by one colour or each of them is coloured by a different colour.

For example, let the three colours be red (R), yellow (Y) and blue (B). The trefoil knot has a projection consisting of three arcs meeting at three crossings. Hence it can easily be coloured by simply drawing each arc by one colour. One could try to colour the figure-8 knot by 3 colours, but it never works. The figure-8 knot is not colourable.

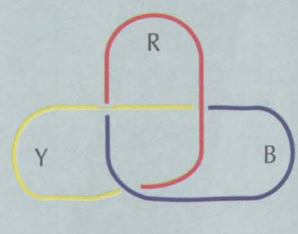


Figure 4

The fact that colourability is a knot invariant is because of the following result.

Theorem

If a diagram of a knot K is colourable, then every diagram of K is colourable.

To prove this result, one verifies that if one diagram of K can be coloured, then the diagram obtained by applying one of the Reidemeister moves to it can also be coloured. This result shows that colourability of knot diagrams is independent of the diagram used to represent the knot. Hence a knot is said to be colourable if one and hence all of its diagrams are colourable. More importantly, if one knot is colourable while another is not, then the two knots are not equivalent. Therefore the trefoil knot and the figure-8 knot are inequivalent.

Let the three colours be denoted by the integers 0, 1, 2. The problem can be reduced to solving a system of linear equations in $Z_3 = \{ 0, 1, 2 \}$ as follows. Let us label the arcs of a given knot diagram of K by the variables x, y, z ... The second condition for colourability is equivalent to the condition that at each crossing the relation $x + y \equiv 2z \pmod 3$ should hold, where z is the label of the overpass and x and y are the labels on the other arcs.

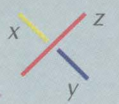


Figure 5

Therefore a knot diagram of K gives rise to a number of linear equations, one for each crossing of the diagram. In this setup, K is colourable if and only if this system of linear equations has a nontrivial solution in Z_3. Note that a trivial solution but not necessarily the zero solution, corresponds to colouring the diagram by just one colour which is not allowed by condition (i). Now the problem reduces to solving a system of homogeneous linear equations in Z_3. As an example, consider the figure 8 knot.

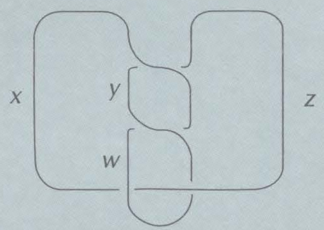


Figure 6

It gives rise to the following system of linear equations.

$$\begin{aligned} 2x - y - z &= 0 \\ -x + 2y - w &= 0 \\ -y + 2z - w &= 0 \\ -x - z + 2w &= 0 \end{aligned}$$

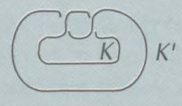
The only solution is $x = y = z = w \pmod 3$. Hence the figure 8 knot is not colourable. In general in a diagram of a knot K, the number of crossings is the same as the number of overpasses. Hence the coefficient matrix of the system of linear equations arising from K is a square matrix. From the theory of linear equations, it is known that such a system has a nontrivial solution in Z_3 if and only if each cofactor of the coefficient matrix is congruent to zero modulo 3. Moreover a nontrival solution in Z_3 corresponds to a colouring of the knot.

Now the generalization to more than 3 colours is immediate. Suppose there are p colours. A knot diagram is said to be p-colourable if the system of homogeneous linear equations posed by the knot diagram has a nontrivial solution in Z_p. As long as p is a prime, the theory of solving system of linear equations is the same as that for real numbers.

Since a knot diagram can be quite arbitrary and complicated, it is not clear when each cofactor of the coefficient matrix is congruent to zero modulo p. It would be useful to obtain equivalent conditions in terms of other properties of the knot. For this, the class of doubled knots is considered in next section.

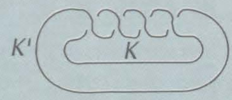
Doubled knots

Let K be a knot and n be an integer. In a knot diagram of K, take a parallel copy K' of K so that K and K' link each other n times. Hence n is the linking number between K and K'. For some details on linking number, see the references [6] and [8]. For example, if K is the unknot, then K' is just another copy of K linking K in full twists.



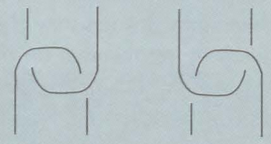
$n = -1$

Figure 7



$n = 2$

Here the twists are right-handed if n is positive and are left-handed if n is negative. To form a doubled knot, remove a small portion of 2 parallel arcs in K and K' and replace it by one of the clasps below.



$\epsilon = -1$ $\quad$ $\epsilon = +1$

Figure 8

The clasps are identified by their signs. The one shown on the left of figure 8 is said to have a clasp sign -1, while the clasp sign of the one on the right is defined to be +1. The following picture shows four doubled knots constructed from the unknot.

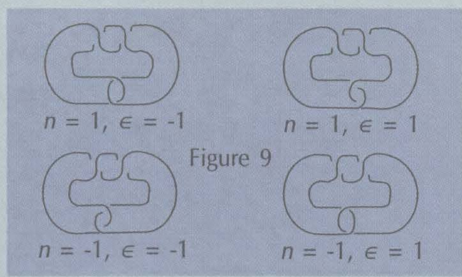


Figure 9

The first and the last are the figure-8 knot. The second and the third are the left-handed trefoil and the right-handed trefoil respectively.

Therefore, given a knot K, an integer n and $\epsilon = \pm 1$, a doubled knot can be constructed as described above. The integer n is often called the framing of the construction. The main result in [8] is the following.

Theorem

Let K be a knot and p be an odd prime. Then the doubled knot formed by K with framing n and with clasp sign ϵ is p-colourable if and only if $4n \equiv \epsilon \pmod p$.

Try it on your friends!

An unusual party trick is performed as follows. Ask spectator A to jot down any three-digit number, and then to repeat the digits in the same order to make a six-digit number (i.e, if A chooses 123, then the six-digit number is 123123). Tell A not to show you the six-digit number. Ask A to pass the six-digit number to spectator B, who is requested to divide the number by 7. Tell B not to worry about the remainder because there would not be any . B is surprised that you are right (e.g. 123123 divided by 7 is 17589). Without telling you the result, B passes it to spectator C who is told to divide it by 11. Again you tell him that there is no remainder (e.g. 17589 divided by 11 is 1599).

With no knowledge whatever of the figures obtained from the computations, you direct a fourth spectator to divide the last result by 13. Again the division comes out without remainders (e.g. 1599 divided by 13 is 123). The final result is written on a slip of paper which is folded and handed to you. Without opening you pass it to A. "Open this," you tell him, "and you will find your original three-digit number."

Question: Why does the trick work?

The proof is constructive. The doubled knot is first divided into small sections. We start by colouring one of the sections. Recall that each colour is regarded as an element in Z_p. A key observation in the proof is that the difference of the colours of the two parallel arcs in each section is a constant modulo p. This enables us to keep track of the changes in colour as we proceed along the knot. When we come to the last section which joins to the starting section, the colours of the connecting ends should match and this gives the condition that

$$4n \equiv \epsilon \pmod{p}.$$

Corollary

A doubled knot is p-colourable for some odd prime p if and only if its framing is non-zero.

As an example, the doubled knot in figure 10 is 3-colourable. Readers may check that the condition in the theorem for this knot is satisfied. An explicit colouring of this knot is also shown in figure 10.

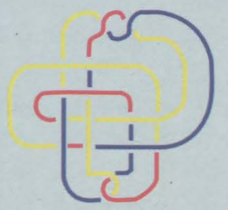


Figure 10

M^2

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This article is based on a project undertaken by Wee Hoe Teck in the Science Research Programme 1994 under the supervision of Dr Wong Yan Loi

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